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Standard and Non-standard Extensions of Lie algebras

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Revised version

Abstract

We study the problem of quadruple extensions of simple Lie algebras. We find that, adding a new simple root α_{+4}, it is not possible to have an extended Kac-Moody algebra described by a Dynkin-Kac diagram with simple links and no loops between the dots, while it is possible if α_{+4} is a Borcherds imaginary simple root. We also comment on the root lattices of these new algebras. The folding procedure is applied to the simply-laced triple extended Lie algebras, obtaining all the non-simply laced ones. Non- standard extension procedures for a class of Lie algebras are proposed. It is shown that the 2-extensions of E_8, with a dot simply linked to the Dynkin-Kac diagram of E_9, are rank 10 subalgebras of E_{10}. Finally the simple root systems of a set of rank 11 subalgebras of E_{11}, containing as sub-algebra E_{10}, are explicitly written.

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1 Introduction

It has been conjectured by Peter West [1] that the still elusive M-Theory possesses a rank eleven Kac-Moody symmetry algebra, called E_{11}, that is the triple extended or very extended E_8 algebra. Very extended algebras can be defined for any finite-dimensional Lie algebra $\mathcal{G}$ [2]. So it is tempting to argue that other theories, which are associated with other triple extensions of Lie algebras, may exist. Indeed the same analysis was applied to a conjectured extension of the eleven dimensional supergravity [3], [4]. More generally, it has been proposed that the closed bosonic string in D dimensions and type I supergravity and pure gravity theories exhibit a Kac-Moody symmetry algebra, respectively identified as the triple extensions of the D and A series [5], [6], [7]. This conjecture is supported by dimensional reduction and by the so called cosmological billiards [8], [9], [10]. Then it is natural to look for a more general symmetry algebra which can include all these Kac-Moody algebras as particular cases. So we address the question of how to go beyond $\mathcal{G}^{+++}$ algebras; we find that the adjoint of a new simple root α_{+4} introduces multiple links and loops in the structure of the 4-extended algebra, if α_{+4} is an ordinary Kac-Moody simple root, while the "simple-links" structure is preserved if we allow α_{+4} is a Borcherds (imaginary) simple root.

The (first) extension of a finite-dimensional Lie algebra is the construction of (untwisted) affine Kac-Moody algebras, which are obtained adding to the simple roots of any finite-dimensional Lie algebra $\mathcal{G}$ a root α_0 that is the opposite of the highest root (h.r.) plus a light-like vector k_+, in order to make α_0 linearly independent from the system of the simple roots of $\mathcal{G}$, keeping unchanged its length, and are denoted as $\mathcal{G}^+$ or $\widehat{\mathcal{G}}$ or $\mathcal{G}^1$. This procedure for the simply laced algebras of the D_N and E_N series can be formulated as the addition to the simple root system of $\mathcal{G}$ of another root α_0, that is the opposite of the unique fundamental weight of length 2, which is in the root lattice of the algebra, plus a light-like vector k_+. The light-like vector can be considered to belong to a 2-dim. Lorentzian lattice, usually denoted $\mathrm{II}^{1,1}$, and the double extension or overextension of $\mathcal{G}$, denoted $\mathcal{G}^{++}$, is obtained by adding a new simple root, of length 2, which is formed by the sum of the two light like vectors $k_\pm$, $(k_+, k_-) = 1$, spanning $\mathrm{II}^{1,1}$. The triple extended or very extended $\mathcal{G}$, denoted $\mathcal{G}^{+++}$, is obtained adding a new simple root of length 2, which belongs to a new copy of the lattice $\mathrm{II}^{1,1}$, plus k_+. In this way, an indefinite Kac-Moody or Lorentzian algebra of rank $r + 3$ is obtained whose roots belong to a Lorentzian lattice of dimension $r + 4$, so it is natural to wonder if an indefinite Kac-Moody algebra of rank $r + 4$ can be obtained by a further extension. Moreover, let us notice that in the lattice $\mathrm{II}^{1,1}$ vectors of negative length do exist (see Appendix A). From this remark one can make an extension of $\mathcal{G}$ adding a new root, that is the opposite of any fundamental weight, that can be written as linear combination with integer coefficients of the simple roots of the algebra, plus a suitable element of the lattice $\mathrm{II}^{1,1}$ in order to have an independent new simple root of length 2. This construction will be discussed below. It has been pointed out in [12] that the structure of subalgebras of hyperbolic Kac-Moody, in general of 2-extended (overextended) Lie algebras, is very rich and surprising. Some of the results of that paper can be generalized to more general extensions and we comment on this point below. This paper is organized as it follows. In Sec. 2, to make the paper self contained, we recall the well-known construction of the 3-extended Lie algebras. We show that the 4-extended algebras are described by Dynkin-Kac diagrams with loops and multiple links (so their structure is quite different from that of the 1-,

2- and 3-extended algebras). In particular, we show that we can not have a situation in which the new (fourth) simple root α_{+4} is simply linked to α_{+3}, unless we let α_{+4} be a Borcherds simple root (with squared norm zero or negative). So we also study the possibility to have a Borcherds extension of the $\mathcal{G}^{+++}$ algebras, but this extension has sense only when the algebra $\mathcal{G}$ is simply-laced (see Appendix B for the definition of a Borcherds algebra [11]). In Sec. 3, we show that all the non-simply laced 3-extended Lie algebras can be obtained by *folding* the simply-laced ones. In Sec. 4, we discuss non standard extension procedures, discussing in detail a few examples which may be of physical interest. In Sec. 5, we show that the algebras obtained by some general non standard procedure, but not for all the procedures, are indeed subalgebras, of the same rank, of the standard triple extended algebras. In particular we prove that any non standard 1-extension of E_9, with a root simply linked to a simple roots of E_9, is a subalgebra of E_{10}. The simple root systems of a set of rank 11 subalgebras of E_{11}, containing as subalgebra E_{10}, are explicitly written. Finally we present a few conclusions and perspectives. To make the paper self-contained two very short Appendices are added to recall the main features of the 2-dim Lorentzian lattice $\mathrm{II}^{1,1}$ and of Borcherds algebras.

2 On extensions of $\mathcal{G}^{+++}$ algebras

An excellent discussion of the mechanism of standard extensions of Lie algebras can be found in [2], here we briefly recall the essential points, mainly to introduce the notation and to make the paper self-consistent [1]. Let $\mathcal{G}$ a simple Lie algebra of rank r, with simple root system α_i ($i = 1, \ldots, r$) and root lattice $\Lambda_{\mathcal{G}} = \bigoplus_{i=1}^{r} \mathbb{Z}\, \alpha_i$ (for the roots and fundamental weights we use the notation of [13]). Let us consider also two copies of the lattice $\mathrm{II}^{1,1}$, which we indicate with $\mathrm{II}^{1,1}_{k_\pm}$ and $\mathrm{II}^{1,1}_{l_\pm}$. In the following, we consider indefinite Kac-Moody algebras with root lattice included in the direct sum $\Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm} \oplus \mathrm{II}^{1,1}_{l_\pm}$.

Let $\mathcal{G}^{+}$ the extended Lie algebra (or affine Kac-Moody algebra), with simple root system $\{\alpha_i$, $\alpha_0 \equiv \alpha_{r+1} \equiv \alpha_{+1} = -h.r. + k_+\}$ ($i = 1, \ldots, r$), where $h.r.$ denotes the highest root of $\mathcal{G}$ (which is $\theta \equiv h.r. = \sum_{i=1}^{r} a_i \alpha_i$ where a_i are Kac marks, see [13]), and

$$(k_+, k_+) = (k_+, \alpha_i) = 0. \tag{1}$$

Let $\mathcal{G}^{++}$ the double extended or overextended Lie algebra (actually a Lorentzian Kac-Moody algebra), with simple root system $\{\alpha_j,\, \alpha_{r+2} \equiv \alpha_{+2} = -(k_+ + k_-)\}$ ($j = 1, \ldots, r+1$), where

$$(k_+, k_-) = 1\,, \qquad (k_-, k_-) = (k_-, \alpha_i) = 0. \tag{2}$$

Let's note that the root lattice of $\mathcal{G}^{+}$ is properly contained in the direct sum $\Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm}$, whereas the root lattice of $\mathcal{G}^{++}$ coincides with the direct sum. To see this, it is enough to obtain k_+ and k_- from the other roots α_j ($j = 1, \ldots, r+2$). In fact,

$$k_+ = \theta + \alpha_{+1} = \sum_{i=1}^{r} a_i\, \alpha_i + \alpha_{+1} \tag{3}$$

[1]Let us remember that the standard extension for G_2 does not work, because G_2 is the only finite Lie algebra for which it is not possible to normalize the highest root θ such that $(\theta, \theta) = 2$. However, it's possible to extend G_2 with the following choice of the extended roots: $\alpha_{+1} = k_+ - \theta$, $\alpha_{+2} = -(k_+ + 3k_-)$ and $\alpha_{+3} = -(l_+ + 3l_-) + k_+$. Anyway, G_2^{+++} can be obtained by folding D_4^{+++} (see Section 3).

$$k_- = -k_+ - \alpha_{+2} = -\theta - \alpha_{+1} - \alpha_{+2} \tag{4}$$

so k_+ and k_- are linear combinations (with integer coefficients) of the $r+2$ simple roots of the $\mathcal{G}^{++}$ algebra. This means that with a change of basis (in the lattice), we can pass from $\{\alpha_j\,(j=1,\ldots,r+2)\}$ to $\{\alpha_i\,(i=1,\ldots,r);\, k_+,\, k_-\}$, that is the lattice $\Lambda_{\mathcal{G}} \oplus \mathbb{Z}\,\alpha_{+1} \oplus \mathbb{Z}\,\alpha_{+2}$ coincides with $\Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm}$.

Now, let $\mathcal{G}^{+++}$ the triple extended Lie algebra (still a Lorentzian Kac-Moody algebra), with simple root system $\{\alpha_k,\, \alpha_{r+3} \equiv \alpha_{+3} = -(l_+ + l_-) + k_+\}$ $(k=1,\ldots,r+2)$, where

$$(\alpha_i, l_\pm) = (k_\pm, l_\pm) = (k_\mp, l_\pm) = (l_\pm, l_\pm) = 0\,, \qquad (l_\mp, l_\pm) = 1. \tag{5}$$

In this way starting from the $r-$dim. Euclidean lattice $\Lambda_{\mathcal{G}}$, we have build up a $(r+4)-$dim. Lorentzian lattice $\mathbf{\Lambda} \equiv \Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm} \oplus \mathrm{II}^{1,1}_{l_\pm}$, with signature $+\ldots+--$ ($r+2$ plus signs). The simple root system of $\mathcal{G}^{+++}$ clearly spans a 3-dim. sub-lattice of $\mathbf{\Lambda}$ (because α_{+3} only takes the direction $l_+ + l_-$ in the second lattice $\mathrm{II}^{1,1}$, so the orthogonal direction is lacking), so it is natural to wonder if it is possible to extend further the $\mathcal{G}^{+++}$ algebra and *fill in* $\mathbf{\Lambda}$.

Motivated by the previous steps, one would just add another node at the Dynkin diagram of a $\mathcal{G}^{+++}$ algebra, with a simple link to the root α_{+3}; in this way, as it happens in the case of the $\mathcal{G}^{++}$ algebra, one would expect that this construction fills the root lattice $\Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm} \oplus \mathrm{II}^{1,1}_{l_\pm}$. In the following, we show that actually this is not the case [2].

As we already mentioned, the simple root α_{+3} contains only the combination $l_+ + l_-$, so if we want to span completely also the second lattice $\mathrm{II}^{1,1}$, we need a new simple root α_{+4} which allows us to obtain l_+ and l_- separately. In this way, we are guaranteed that the root lattice of the new algebra contains also all vectors which are integer multiples of l_+ or l_- , so this lattice coincides with $\mathbf{\Lambda}$. In the following, let's choose for α_{+4} the more general form:

$$\alpha_{+4} \equiv a\, l_+ + b\, l_- + c\, k_+ + d\, k_- \tag{6}$$

which allows us to make many considerations about the possible extensions of the $\mathcal{G}^{+++}$ algebras [3]. First of all, let's obtain the expression for l_+ and l_- using the definition of α_{+3} and α_{+4}:

$$(a-b)\, l_+ = (d-b-c)\,\theta + (d-b-c)\,\alpha_{+1} + d\,\alpha_{+2} + b\,\alpha_{+3} + \alpha_{+4}\ , \tag{7}$$

$$(b-a)\, l_- = (d-a-c)\,\theta + (d-a-c)\,\alpha_{+1} + d\,\alpha_{+2} + a\,\alpha_{+3} + \alpha_{+4}. \tag{8}$$

We see then that it is possible to obtain l_+ and l_- as linear combinations (over $\mathbb{Z}$) of the simple roots if and only if $|\,a-b\,| = 1$. Only in this case we are guaranteed that we actually catch all the vectors in the lattice $\mathrm{II}^{1,1}_{l_\pm}$. With this in mind, let's find the other relations a, b, c, d have to satisfy, that is let us compute the norm of α_{+4} and the scalar products with the other simple roots (here we write only the relevant elements a_{ij} of the generalized Cartan matrix):

$$(\alpha_{+4})^2 = 2\,(a\,b + c\,d)\ , \tag{9}$$

[2]Our discussion does not take account for the introduction of another lattice $\mathrm{II}^{1,1}$, which solves the question only partially, because it moves the problem to fill this new lattice.

[3]As it happens also for α_{+2} and α_{+3}, here we do not consider the situation in which α_{+4} is also linked with the simple roots α_i of $\mathcal{G}$. However this simple choice does not really change the conclusions of our analysis .

$$a_{+4,+1} \equiv 2\,\frac{(\alpha_{+4},\alpha_{+1})}{(\alpha_{+4},\alpha_{+4})} = \frac{d}{a\,b+c\,d}\,, \qquad a_{+4,+2} \equiv 2\,\frac{(\alpha_{+4},\alpha_{+2})}{(\alpha_{+4},\alpha_{+4})} = \frac{-c-d}{a\,b+c\,d}\,, \tag{10}$$

$$a_{+4,+3} \equiv 2\,\frac{(\alpha_{+4},\alpha_{+3})}{(\alpha_{+4},\alpha_{+4})} = \frac{d-a-b}{a\,b+c\,d}\,. \tag{11}$$

Let us limit ourselves to consider the case $(\alpha_{+4})^2 = 2$ (to look for a "natural extension") and let all the four coefficients be different from zero; we must have:

$$a\,b + c\,d = 1\,, \tag{12}$$

$$d < 0\,, \qquad c \geq -d > 0\,, \qquad d \leq a+b\,, \tag{13}$$

because only with these constraints α_{+4} is an acceptable simple root (à la Kac-Moody). In particular, the product $c\,d$ must be negative, and as eq. (12) holds, the product $a\,b$ must be positive (actually, it must be at least 2), that is a and b must have the same sign. A simple solution [4] to eqs. (12)-(13) is the following:

$$a = 2, \quad b = 1, \quad c = 1, \quad d = -1, \tag{14}$$

where α_{+4} has norm 2 and the scalar products are $(\alpha_{+4},\alpha_{+1}) = -1$, $(\alpha_{+4},\alpha_{+2}) = 0$ and $(\alpha_{+4},\alpha_{+3}) = -4$. Furthermore, this solution verifies also the condition $a - b = 1$, so its root lattice is precisely $\mathbf{\Lambda}$, but it presents one loop and a multiple link with the (last) root α_{+3}, so its structure is very different from the other previous extensions [5]. This situation is common to all solutions of eqs. (12)-(13): it is not possible (we stress: without adding any $\mathrm{II}^{1,1}$ or different lattice) to have a simple link with α_{+4} and no loops (for example the equation $d - a - b = -1$ is never satisfied if a and b are both positive, because d is at least -1, and goes in contradiction with the other relations if a and b are both negative). In this sense, the procedure of standard extension stops at the third step ($\mathcal{G}^{+++}$).
It is clear that there exist infinite solutions to eqs. (12)-(13) (for example, it is enough to take a, b both positive, $d = -1$ and an opportune value of c to satisfy eq. (12) and the second of eqs. (13)), but only those with $|a - b| = 1$ have the property that their root lattice coincides with $\mathbf{\Lambda}$. We can also try (in α_{+4}) to put one coefficient equal to zero and to explore more specific cases. An investigation, case by case, shows that:

- $d = 0$: the scalar products imply $ab = 1$ and $a + b > 0$, that is $a = b = 1$, but the case $a = b$ is not acceptable because, in this case, α_{+4} is a linear combination of the other simple roots;

- $c = 0$: the scalar products with α_{+1} and α_{+2} are respectively proportional to d and $-d$, so one of the two is positive and has the wrong sign, or $d = 0$ and then α_{+4} is a linear combination of the other simple roots;

- $a = 0$: this case implies $c = -1$, but then $d = -1$ too and the scalar product with α_{+2} is positive;

[4] We thank C. Helfgott for pointing us this solution.

[5] We thank A. Kleinschmidt for suggesting us another solution $\alpha_{+4} = \alpha + 3\,k_+ - 2\,k_- - 2\,l_+ - 3\,l_-$ in which α_{+4} is also linked to the only simple root α of su(2); this solution too presents loops and multiple links.

- $b = 0$: it has the same problem as the case $a = 0$.

If we put equal to zero two coefficients, the only possibility is to have $c = d = 0$ (we can not take equal to zero a coefficient of l_+ or l_- and a coefficient of k_+ or k_-, because α_{+4} would have vanishing norm), but in this case, as already discussed, we have problems.
So far, we have seen that it is never possible to add a (Kac-Moody) simple root in order to have a simple link with α_{+3}; besides this, the possible solutions which span the whole lattice $\mathbf{\Lambda}$ are restricted to the condition $|a - b| = 1$. Yet, if we abandon the condition that α_{+4} is a Kac-Moody simple root and let it be a Borcherds (imaginary) simple root, the situation changes. In fact, we can easily find an imaginary root of the kind:

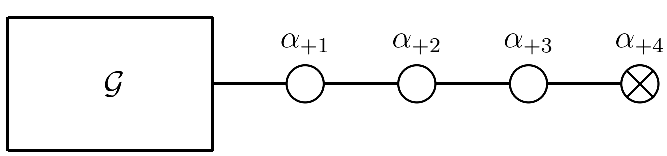
$$\alpha_{+4} = a\, l_+ + b\, l_- \tag{15}$$

which has scalar product equal to -1 with α_{+3}: it is only necessary that $b = 1 - a$, that is $\alpha_{+4} = a\, l_+ + (1 - a)\, l_-$, with norm $2\, a\,(1 - a)$. Then if $a = 0$ or $a = 1$, α_{+4} has norm zero, in all the other cases its norm is negative, that is α_{+4} is a good Borcherds simple root. If we want to fill in $\mathbf{\Lambda}$ with the introduction of a Borcherds simple root, we have always to fulfill the condition $|a - b| = 1$ (this condition is independent from the norm of α_{+4}), but now it is possible to have both $(\alpha_{+4}, \alpha_{+3}) = -1$ and, at the same time, to span the whole lattice $\mathbf{\Lambda}$. In fact the values $a = 1, b = 0$ (and viceversa $a = 0, b = 1$), which correspond to $\alpha_{+4} = l_+$ ($\alpha_{+4} = l_-$), are the only ones which allow to have:

$$\Lambda_{roots}(\mathcal{BG}^{+++}) \equiv \Lambda_{roots}(\mathcal{G}^{++}) \oplus \mathrm{II}^{1,1}_{l_\pm} = \mathbf{\Lambda}\ , \tag{16}$$

$$(\alpha_{+4}, \alpha_{+3}) = -1\, , \tag{17}$$

as opposite to the Kac-Moody case, where, as we already said, it's never possible to satisfy eq. (17) and only some solutions allow to fill in $\mathbf{\Lambda}$ (we have called $\mathcal{BG}^{+++}$ the Borcherds extension of $\mathcal{G}^{+++}$ corresponding to $\alpha_{+4} = l_+$ or $\alpha_{+4} = l_-$). Denoting by a crossed dot the Borcherds simple root α_{+4}, we can draw the Dynkin diagram of this Borcherds algebra in the following way:



Actually, this construction makes sense if $\mathcal{G}$ is simply-laced, because otherwise the Cartan matrix is not well-defined. In fact, if α_{+4} is an imaginary (isotropic) root, we cannot define the extended Cartan matrix as $2\,\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ for all α_i because $\alpha_{+4}^2 = 0$. We have the same problem if we add an imaginary simple root with negative squared norm (as previously recalled, in $\mathrm{II}^{1,1}$ there are infinite vectors whose squared norm is negative), because we shall have positive elements out of the principal diagonal. The solution is then to consider a $\mathcal{G}$ simply-laced and define the extended Cartan matrix by the scalar products between all the simple roots: $b_{i,j} := (\alpha_i, \alpha_j)$. So, while the extension of $\mathcal{G}$ is possible up to $\mathcal{G}^{+++}$ for any finite $\mathcal{G}$, in the case of a Borcherds extension it is necessary to choose for $\mathcal{G}$ a simply-laced algebra (in fact, a Borcherds algebra is defined only on a symmetric Cartan matrix). To summarize our result, we have proven the following

Proposition 1 *The extension of $\mathcal{G}^{+++}$ algebra, whose simple root system spans completely $\mathbf{\Lambda}$ and whose Dynkin-Kac diagram has no loops and only simple links between the dots, is the Borcherds algebra $\mathcal{BG}^{+++}$ (with $\mathcal{G}$ simply-laced).*

The particular solution $a = 1, b = 0$ (or viceversa) looks like the same construction of $\mathcal{G}^{+}$ and $\mathcal{G}^{++}$, as the role of $k_\pm$ is now played by $l_\pm$, with the difference that α_{+1} contains also a root of a simple Lie algebra, while α_{+4} doesn't. This observation suggests that it's possible to fuse together two (finite) simple Lie algebras, let's say $\mathcal{G}$ and $\mathcal{G}'$, adding the highest root θ' of $\mathcal{G}'$ to α_{+4}. In this way, $\alpha'_{+4} \equiv \alpha_{+4} - \theta'$ is again a Kac-Moody simple root, indeed the affine root of $\mathcal{G}'^{+}$. Anyway, we don't insist on this point, because there are many possible ways to fuse together two (or more) finite dimensional simple Lie algebras (with or without the introduction of intermediate Kac-Moody or Borcherds algebras).

Remark: In this section, we have seen that triple extended Lie algebras $\mathcal{G}^{+++}$ have their root lattice properly included in $\mathbf{\Lambda} = \Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1} \oplus \mathrm{II}^{1,1}$ (in particular, their root lattice is Lorentzian with just one negative eigenvalue). Among the 4-extensions we considered in this Section, we have shown that there are many algebras (of Kac-Moody or generalized Kac-Moody type) whose root lattice coincides with $\mathbf{\Lambda}$, which is Lorentzian in a more general sense (it has two negative eigenvalues). These seem to be the first algebras obtained in literature with this kind of lattice, together with some similar algebras studied by Harvey and Moore in [20], [21] (strictly speaking, their algebras are not Borcherds algebras, because they could not satisfy some grading conditions in the characterization of generalized Kac-Moody algebras, while our Borcherds algebras are *true* Borcherds algebras because we have constructed them from an acceptable generalized Cartan matrix). Our result seems to be in contrast with the statement of [22], according to which algebras whose root lattices are of the kind $\Gamma^{p,q}$ (that is, with a signature with more that one negative sign) cannot be described in terms of generators and relations similar to Kac-Moody or Borcherds algebras and belong to a new class of Lie algebras. That result has been found looking for Lie algebras of the physical states of a vertex algebra constructed on a general even self-dual lattice $\Gamma^{p,q}$; maybe the condition of physical states is too strong and is not compatible with the algebras constructed by us.

3 Folding of triple extended Lie algebras

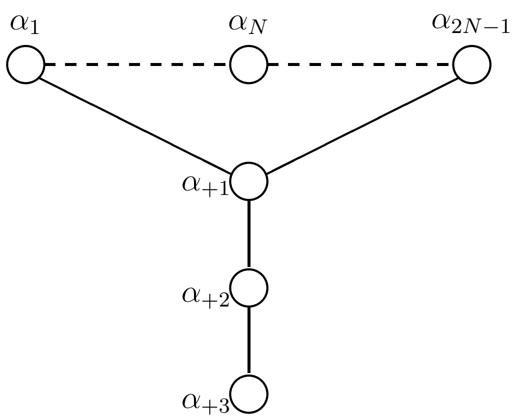
The *folding* technique is a simple and powerful method to find class of singular subalgebras of finite Lie algebras as well as of affine or indefinite Kac-Moody algebras. The starting point is to use the symmetry τ of the Dynkin diagram, corresponding to an exterior automorphism of the algebra $\mathcal{G}$. In the finite case all the Dynkin diagrams of simply-laced algebras show an automorphism of order $k = 2$, except the case of D_4 where the order is 3. Let α_i be a simple root of $\mathcal{G}$. Using the automorphism τ of order k, we obtain

$$\beta_i := \alpha_i + \tau(\alpha_i) + \cdots + \tau^{k-1}(\alpha_i) \tag{18}$$

which form the simple root system of a singular subalgebra $\mathcal{H}$ of $\mathcal{G}$. The generators of $\mathcal{G}$, corresponding to the simple roots α_i, left unchanged by τ, become generators of $\mathcal{H}$, while the other ones transform according to a relation analogous to eq.(18). In the following we apply the folding method to the triple extended Lie algebras, obtaining all the non simply laced triple extended algebras, as in the finite case. The automorphism of the 3-extended Dynkin diagram

acts on the standard way upon the roots of the finite classical subalgebras and trivially on the extended roots $\alpha_{+1}, \alpha_{+2}, \alpha_{+3}$. This property has to hold if we want to preserve the structure of the triple extension of the non-simply laced algebras [6]. Let us enumerate all the cases.

3.1 $\mathbf{A}^{+++}_{2N-1} \longrightarrow \mathbf{C}^{+++}_{N}$



The Cartan matrix of A^{+++}_{2N-1} can be written, in block form, as

$$A = (a_{ij}) = (\alpha_i, \alpha_j) = \left(\begin{array}{ccc|ccc} & & & -1 & 0 & 0 \\ & A_{A_{2N-1}} & & \vdots & \vdots & \vdots \\ & & & -1 & 0 & 0 \\ -1 & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{array} \right) \qquad (19)$$

The not trivial action of τ on the simple roots gives

$$\begin{aligned} \beta_1 &:= \alpha_1 + \tau(\alpha_1) = \alpha_1 + \alpha_{2N-1} & (20) \\ \beta_2 &:= \alpha_2 + \tau(\alpha_2) = \alpha_2 + \alpha_{2N-2} \quad \ldots & (21) \\ \beta_{N-1} &:= \alpha_{N-1} + \tau(\alpha_{N-1}) = \alpha_{N-1} + \alpha_{N+1} & (22) \\ \beta_N &:= \alpha_N + \tau(\alpha_N) = 2\alpha_N & (23) \\ \beta_{+1} &:= \alpha_{+1} + \tau(\alpha_{+1}) = 2\alpha_{+1} & (24) \\ \beta_{+2} &:= \alpha_{+2} + \tau(\alpha_{+2}) = 2\alpha_{+2} & (25) \\ \beta_{+3} &:= \alpha_{+3} + \tau(\alpha_{+3}) = 2\alpha_{+3} & (26) \end{aligned}$$

the length of the simple roots is

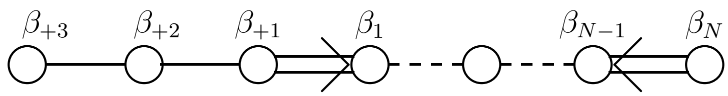
$$\begin{aligned} \beta_1^2 &= \ldots = \beta_{N-1}^2 = 4 & (27) \\ \beta_{+3}^2 &= \beta_{+2}^2 = \beta_{+1}^2 = \beta_N^2 = 8\,. & (28) \end{aligned}$$

[6] Indeed, also the non simply-laced G^{++} algebras can be obtained with the same folding technique from the simply-laced $\mathcal{G}^{++}$, while a different kind of folding applied to the $\mathcal{G}^{+}$ algebras allows to get all the twisted affine algebras.

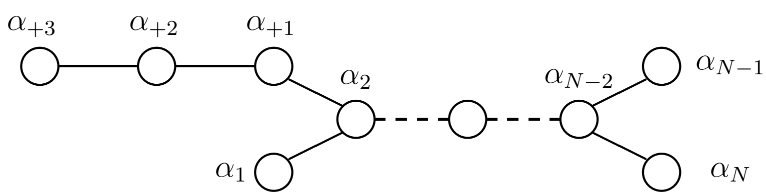
The corresponding Cartan matrix is

$$B = (b_{ij})_{i,j} = 2\frac{(\beta_i,\beta_j)}{(\beta_i,\beta_i)} = \begin{pmatrix} & & & -2 & 0 & 0 \\ & A_{C_N} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ -1 & \cdots & 0 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \tag{29}$$

So we get the 3-extended Lie algebra C_N^{+++} with Dynkin diagram



3.2 $\mathbf{D}_N^{+++} \longrightarrow \mathbf{B}_{N-1}^{+++}$



The Cartan matrix can be written as

$$A = (a_{ij})_{i,j} = (\alpha_i,\alpha_j) = \begin{pmatrix} & & & & 0 & 0 & 0 \\ & & & & -1 & \vdots & \vdots \\ & & A_{D_N} & & \vdots & \vdots & \vdots \\ & & & & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \tag{30}$$

The non trivial action of τ is

$$\tau(\alpha_{N-1}) = \alpha_N, \quad \tau(\alpha_N) = \alpha_{N-1} \tag{31}$$

The new simple roots are

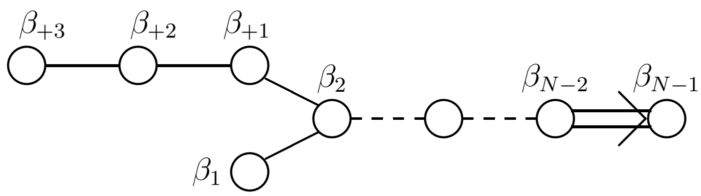
$$\begin{aligned} \beta_{N-1} &:= \alpha_{N-1} + \tau(\alpha_{N-1}) = \alpha_{N-1} + \alpha_N & (32) \\ \beta_{N-2} &:= \alpha_{N-2} + \tau(\alpha_{N-2}) = 2\alpha_{N-2} \quad \ldots & (33) \\ \beta_1 &:= \alpha_1 + \tau(\alpha_1) = 2\alpha_1 & (34) \\ \beta_{+1} &:= \alpha_{+1} + \tau(\alpha_{+1}) = 2\alpha_{+1} & (35) \\ \beta_{+2} &:= \alpha_{+2} + \tau(\alpha_{+2}) = 2\alpha_{+2} & (36) \\ \beta_{+3} &:= \alpha_{+3} + \tau(\alpha_{+3}) = 2\alpha_{+3} & (37) \end{aligned}$$

where

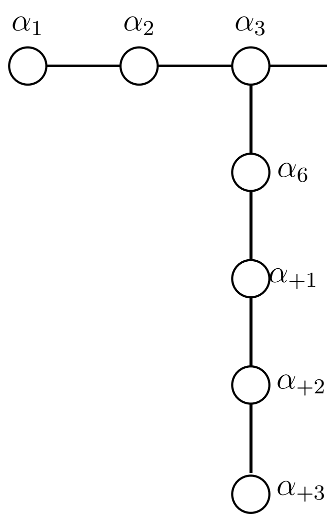
$$\beta_{N-1}^2 = 4\,, \qquad \beta_{+3}^2 = \beta_{+2}^2 = \ldots = \beta_{N-2}^2 = 8\,. \tag{38}$$

The Cartan matrix and the corresponding Dynkin diagram of B_{N-1}^{+++} are

$$B = (b_{ij})_{i,j} = 2\,\frac{(\beta_i,\beta_j)}{(\beta_i,\beta_i)} = \left(\begin{array}{ccccccc} & & & & 0 & 0 & 0 \\ & & & & -1 & \vdots & \vdots \\ & & A_{B_{N-1}} & & \vdots & \vdots & \vdots \\ & & & & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 2 \end{array}\right) \tag{39}$$



3.3 $\mathbf{E}_6^{+++} \longrightarrow \mathbf{F}_4^{+++}$



The Cartan matrix of E_6^{+++} is

$$A = (a_{ij})_{i,j} = (\alpha_i,\alpha_j) = \left(\begin{array}{cccccc} & & & 0 & 0 & 0 \\ & A_{E_6} & & \vdots & \vdots & \vdots \\ & & & -1 & 0 & 0 \\ 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{array}\right) \tag{40}$$

The simple roots are given by

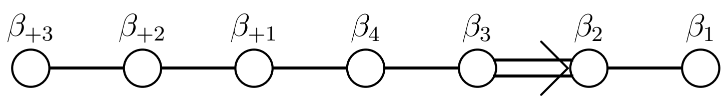
$$\beta_1 := \alpha_1 + \tau(\alpha_1) = \alpha_1 + \alpha_5 \tag{41}$$
$$\beta_2 := \alpha_2 + \tau(\alpha_2) = \alpha_2 + \alpha_4 \tag{42}$$
$$\beta_3 := \alpha_3 + \tau(\alpha_3) = 2\alpha_3 \tag{43}$$
$$\beta_4 := \alpha_6 + \tau(\alpha_6) = 2\alpha_6 \tag{44}$$
$$\beta_{+1} := 2\alpha_{+1}, \quad \beta_{+2} := 2\alpha_{+2} \tag{45}$$
$$\beta_{+3} := 2\alpha_{+3} \tag{46}$$

and

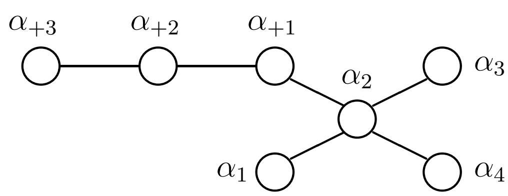
$$\beta_1^2 = \beta_2^2 = 4\,, \qquad \beta_{+3}^2 = \beta_{+2}^2 = \ldots = \beta_3^2 = 8\,. \tag{47}$$

One gets the 3-extended algebra F_4^{+++}, with Cartan matrix and Dynkin diagram

$$B = (b_{ij})_{i,j} = 2\,\frac{(\beta_i,\beta_j)}{(\beta_i,\beta_i)} = \left(\begin{array}{cccccc} & & & 0 & 0 & 0 \\ & A_{F_4} & & \vdots & \vdots & \vdots \\ & & & -1 & 0 & 0 \\ 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{array}\right) \tag{48}$$



3.4 $D_4^{+++} \longrightarrow G_2^{+++}$



The Cartan matrix of D_4^{+++} is

$$A = (a_{ij})_{i,j} = (\alpha_i,\alpha_j) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{49}$$

The action of τ on the simple roots gives

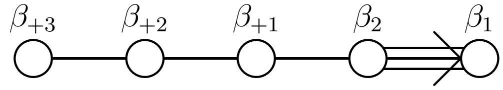
$$\beta_1 \;:=\; \alpha_1 + \tau(\alpha_1) + \tau^2(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 \tag{50}$$
$$\beta_2 \;:=\; 3\alpha_2 \tag{51}$$
$$\beta_{+1} \;:=\; 3\,\alpha_{+1}\,, \quad \beta_{+2} := 3\,\alpha_{+2} \tag{52}$$
$$\beta_{+3} \;:=\; 3\,\alpha_{+3} \tag{53}$$

and

$$\beta_1^2 = 6\,, \qquad \beta_{+3}^2 = \ldots = \beta_2^2 = 18\,. \tag{54}$$

One gets the extended algebra G_2^{+++} with Cartan matrix and Dynkin diagram:

$$B = (b_{ij})_{i,j} = 2\,\frac{(\beta_i,\beta_j)}{(\beta_i,\beta_i)} = \begin{pmatrix} 2 & -3 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{55}$$



Let us remark that, in general, by applying the folding procedure to an algebra $\mathcal{G}$, defined by the Cartan matrix A and Dynkin diagram $S(A)$, we obtain the Cartan matrix B and the corresponding Dynkin diagram $S(B)$ of another algebra $\mathcal{H}$. However we have to check that the new generators, defined in function of the generators of of $\mathcal{G}$, satisfy all the defining relations of the algebra $\mathcal{H}$. Let us see how the new generators are obtained. Let α_i be a simple root of of $\mathcal{G}$ and h_i, e_i, f_i the associated generators. Let us denote by β_i, respectively h'_i, e'_i, f'_i, the roots and the associated generators transformed under the action of τ, which we identify, respectively, as the simple roots and the associated generators of $\mathcal{H}$. If the action of τ is trivial, that is $\beta_i = k\alpha_i$ (where k is the order of the automorphism τ, $\tau^k = 1$) then the generators are not transformed $h'_i = h_i$, $e'_i = e_i$ and $f'_i = f_i$. If the action of τ is not trivial, that is $\beta_i = \alpha_i + \tau(\alpha_i) + \ldots + \tau^{k-1}(\alpha_i)$, we obtain $h'_i = h_i + h_{\tau(\alpha_i)} + \ldots + h_{\tau^{k-1}(\alpha_i)}$, $e'_i = e_i + e_{\tau(\alpha_i)} + \ldots + e_{\tau^{k-1}(\alpha_i)}$ and $f'_i = f_i + f_{\tau(\alpha_i)} + \ldots + f_{\tau^{k-1}(\alpha_i)}$. We have to verify that the generators h'_i, e'_i, f'_i satisfy the defining relations

$$[e'_i, f'_j] = \delta_{ij}\, h'_j,$$

$$[h'_i, e'_j] = b_{ij}\, e'_j, \quad [h'_i, f'_j] = -b_{ij}\, f'_j,$$

$$[h'_i, h'_j] = 0,$$

$$(\mathrm{ad}\, e'_i)^{1-b_{ij}}\, e'_j = 0, \quad (\mathrm{ad}\, f'_i)^{1-b_{ij}}\, f'_j = 0,\ (i \neq j).$$

We do no report here the explicit calculations, but everything works nicely. Finally it should be remarked that the folding procedure for indefinite Kac-Moody algebra, when applicable, always gives rise to indefinite Kac-Moody algebra, as it happens for the finite, affine, hyperbolic Kac-Moody algebras. On the contrary other reduction procedures, as the orbifolding, do not preserve the kind of algebras. Indeed as remarked in [19], the orbifolding of E_{10}, to which the folding procedure cannot be applied, gives rise to non Kac-Moody algebras.

4 Non-standard extensions of Lie algebras

In this section we present a non-standard construction of extended Lie algebras; as stated in Sec. 1, the idea of the non-standard extension is to add to the simple root system $\{\alpha_i\}$ of a simple Lie algebra $\mathcal{G}$ new roots, which are formed by those fundamental weights of the algebra that are linear combinations with integer coefficients of α_i, plus a suitable combinations of vectors belonging to the Lorentzian lattice $\mathrm{II}^{1,1}_{k\pm}$ and/or $\mathrm{II}^{1,1}_{l\pm}$. The new roots have to satisfy the requirements that their squared norms are equal to 2 and that are suitably linked with the previous ones. Let us remark that the roots of the non-standard extension do not generally span the whole lattice $\mathbf{\Lambda} = \Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k\pm} \oplus \mathrm{II}^{1,1}_{l\pm}$ and that, moreover, the structure of the added simple root is, by no way, unique. Of course one can add more than two 2-dim. Lorentzian lattices, but these extensions will not be considered in the present paper, where we add at most three new roots. Also we shall not discuss the case where the squared norm of the added roots is not equal to 2. So, given a simple Lie algebra $\mathcal{G}$, we add to the root lattice $\Lambda_{\mathcal{G}}$ a new simple root $\alpha_{r+1} \equiv \alpha_{+1}$, which is formed by the opposite of a fundamental weight $-\Lambda_i$ and

by a suitable linear combination, with integer coefficients, of the vectors $k_\pm$, in order to have $\alpha_{+1}^2 = 2$, as Λ_i^2 is not necessarily 2. Let's remember that the fundamental weights have the property $2\frac{(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{i,j}$; they span the weight lattice $P = \bigoplus_{i=1}^r \mathbb{Z}\Lambda_i$, which is dual to the coroot lattice $\Lambda_{\mathcal{G}}^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ where $\alpha_i^\vee = 2\frac{\alpha_i}{(\alpha_i,\alpha_i)}$ are the coroots. So while it is always true that $P = (\Lambda_{\mathcal{G}}^\vee)^*$, in general we have $\Lambda_{\mathcal{G}} \subseteq P$. This means that for each $\mathcal{G}$, only some Λ_i belong to the root lattice; so, in defining α_{+1} we choose the $\Lambda_i \in \Lambda_{\mathcal{G}}$. Since Λ_i is only linked with the simple root α_i, we have $(\alpha_{+1}, \alpha_j) = -\delta_{+1,\,i}$. So the first extension is made by adding the root $\alpha_{r+1} \equiv \alpha_{+1} \equiv := -\Lambda_i + k_+ - ak_-$, where $a \in \mathbf{Z}_+$ is fixed by the condition $\alpha_{+1}^2 = \Lambda_i^2 - 2a = 2$. At this point, we add the simple root $\alpha_{r+2} \equiv \alpha_{+2} := -\theta + bk_- - l_-$, where θ is the highest root of $\mathcal{G}$ and $b \in \mathbf{Z}$ is a coefficient chosen in order to have $(\alpha_{+2}, \alpha_{+1}) = (\Lambda_i,\, \theta) - b = 0$. In this way, we have $\alpha_{+2}^2 = 2$ and α_{+2} behaves like an affine root (that is, it is linked with the simple roots of $\mathcal{G}$ in a way completely analogous as the affine root of the algebras $\widehat{\mathcal{G}}$). At the end, we add the third simple root $\alpha_{r+3} \equiv \alpha_{+3} := l_+ + l_-$ with the property that $(\alpha_{+3}, \alpha_{+2}) = -1$ and $(\alpha_{+3}, \alpha_i) = 0$ for $i = 1, \ldots, r, r+1$. As $(\Lambda_i,\, \Lambda_j) \in \mathbf{Z}_>$ for the Lie algebra $\mathcal{G}$ below considered, this procedure is completely general and the extended algebra contains as subalgebra the affine extension of $\mathcal{G}$ (so sometimes we shall call this a *non standard affine extension*). Clearly the lightlike vector l_- can be hanged up to any other simple root, producing an other indefinite Kac-Moody. We shall comment on this point in Sec.5. This construction leads to indefinite Kac-Moody algebras, 1-, 2- and 3-extended, whose (symmetric) Cartan matrix $2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ (for $i, j = 1, \ldots, r+3$) has Lorentzian signature $(+\ldots+-)$ with $r+2$ plus signs and 1 minus sign. The root lattice of the 3-extended algebra is properly contained in $\Lambda_{\mathcal{G}} \oplus \mathrm{II}^{1,1}_{k_\pm} \oplus \mathrm{II}^{1,1}_{l_\pm}$. For these algebras, one can make similar discussions as those in Sec. 2 on eventual further extensions. Now we want to discuss another possible extension, which cannot be performed for any fundamental weight Λ_i belonging to the root lattice of $\mathcal{G}$. The first extension is performed as before, but as second extension we add the root $\alpha_{+2} := -\Lambda_j + k_+ - bk_- - l_-$ $(i \neq j)$, where $b \in \mathbf{Z}_+$ is such that $\alpha_{+2}^2 = 2$ and $(\alpha_{+1}, \alpha_{+2}) = (\Lambda_i, \Lambda_j) - a - b = 0$. Below we shall show that, for $\mathcal{G} \neq E_6$, for any i $(\Lambda_i \neq \theta)$, at least one j exists which satisfies the above condition. In the following, we discuss only some examples of the general construction, we called *affine extension*; in particular we concentrate on the simply laced algebras, but it is possible to consider also the other cases paying attention at the choice of the fundamental weight.

Looking at the fundamental weights of simply laced-Lie algebras, see [13], one realizes that the fundamental weights which can be written as

$$\Lambda_i = \sum_n c_n\, \alpha_n \qquad c_n \in \mathbf{Z} \tag{56}$$

are

1. for $D_N = so(2N)$ $(N \geq 4)$, the weights Λ_i with i even number $(N-2 \geq i \geq 2)$
2. for E_6, only the weights Λ_i $(i = 3, 6)$
3. for E_7, only the weights Λ_i $(i = 1, 2, 3, 5)$
4. for E_8, all the weights Λ_i, which is just a consequence of the E_8-lattice being a self-dual one.

In the following we discuss some of the possible non-standard extensions, with the aim to illustrate the procedure in a few examples which may be relevant for their subalgebras content. Let us emphasize that the discussed extensions as well their subalgebras content are not at all exhaustive, being the choice of the extended simple roots not unique, in general.

4.1 $D_N = so(2N)$

In order to illustrate the general procedure, we discuss in some detail the case of $D_6 = so(12)$, which is the first algebra of the even orthogonal series which admits a non-standard extension. We add the simple root

$$\alpha_{+1} := -\Lambda_4 + k_+ - k_-, \qquad \alpha_{+1}^2 = 2, \tag{57}$$

where Λ_4 is the fundamental weight [7]

$$\Lambda_4 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \qquad \Lambda_3^2 = 4, \tag{58}$$

Clearly we have

$$(\alpha_{+1}, \alpha_i) = -\delta_{4,\,i}. \tag{59}$$

We add now the root

$$\alpha_{+2} := -h.r. - 2k_- - l_- = -(\varepsilon_1 + \varepsilon_2) - 2k_- - l_-, \tag{60}$$

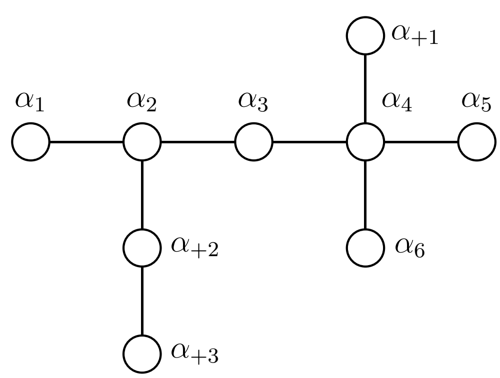
$$(\alpha_{+2}, \alpha_j) = -\delta_{2,\,j}, \tag{61}$$

and

$$\alpha_{+3} := l_+ + l_-, \tag{62}$$

$$(\alpha_{+3}, \alpha_k) = -\delta_{+2,k}, \tag{63}$$

with Dynkin diagram:



Let's observe that the first extension of D_6 is the same algebra as D_4^{+++}, so folding α_{+1}, α_5 and α_6 we re-obtain G_2^{+++}. Clearly the choice of the extended simple roots is not unique. One can easily see that:

[7]the ε_i are unit ortho-normal vectors in $\mathbb{R}^6$.

- a non-standard extension of $so(4N)$ admits as subalgebra the affine extension of $so(4N)$ and $so(4(N-1))$. Indeed one adds to the roots of $so(4N)$ the root of the affine extension

$$\alpha_{+1} := -\Lambda_2 + k_+ \tag{64}$$

and the new non-standard root

$$\alpha_{+2} := -\Lambda_4 + l_+ - l_- \tag{65}$$

Taking away the roots α_{+1}, α_j $(j = 1, 2)$ one gets the algebra $so(4(N-1))^{(1)}$. Let us remark that if we add the root

$$\alpha_{+2} := -\Lambda_6 + l_+ - 2l_- \tag{66}$$

and then we take away the roots α_{+1}, α_j $(j = 1, \ldots, 4)$ one gets the algebra $so(4(N-2))^{(1)}$.

- the non standard extension of $so(24)$ is the smallest extension of the orthogonal series which contains as subalgebra E_{11}. Indeed adding to the roots of $so(24)$ the non standard root

$$\alpha_{+1} := -\Lambda_8 + k_+ - 3k_- \tag{67}$$

and deleting α_{11}, α_{12} one gets E_{11}.

Let us call $\widehat{\Lambda}_{2n} = -\Lambda_{2n} + k_+ - (n-1)k_-$, where $n \in \mathbb{Z}_+$ and $\Lambda_{2n} = \sum_{i=1}^{2n} \varepsilon_i$ is a fundamental weight. Clearly we have

$$\widehat{\Lambda}_{2n}^2 = 2 \qquad (\widehat{\Lambda}_{2n}, \widehat{\Lambda}_{2n+2}) = 0 \tag{68}$$

4.2 E_6

Let us add to the simple root system of E_6 the root

$$\alpha_{+1} := -\Lambda_3 + k_+ - 2k_-, \qquad \alpha_{+1}^2 = 2, \tag{69}$$

where Λ_3 is the fundamental weight

$$\Lambda_3 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_8 - \varepsilon_7 - \varepsilon_6, \qquad \Lambda_3^2 = 6. \tag{70}$$

Clearly we have

$$(\alpha_{+1}, \alpha_i) = -\delta_{3,\,i}. \tag{71}$$

We add now the root

$$\alpha_{+2} := -h.r. - 3k_- - l_- = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8) - 3k_- - l_-, \tag{72}$$

$$(\alpha_{+2}, \alpha_j) = -\delta_{6,\,j}, \tag{73}$$

and

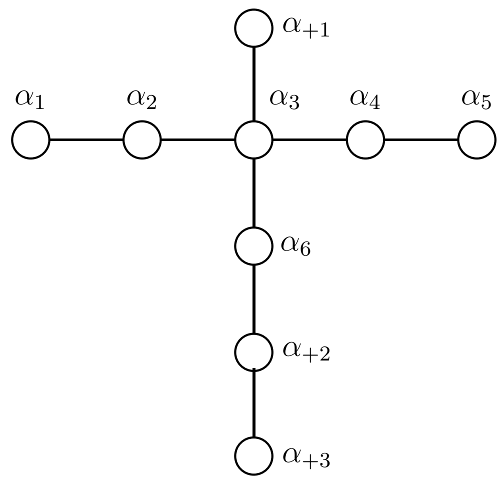
$$\alpha_{+3} := l_+ + l_-, \qquad (\alpha_{+3}, \alpha_k) = -\delta_{+2,\,k}. \tag{74}$$

Alternatively we can add the roots

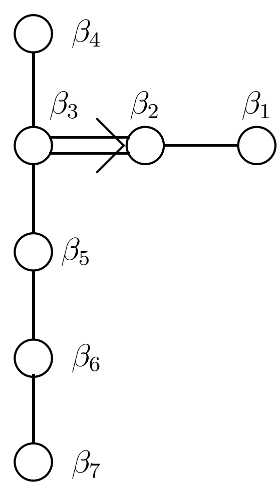
$$\alpha_{+2} \;:=\; k_- + k_+ - l_-, \qquad (\alpha_{+2}, \alpha_j) = -\delta_{+1,\,j}, \tag{75}$$

$$\alpha_{+3} \;:=\; l_+ + l_-, \qquad (\alpha_{+3}, \alpha_k) = -\delta_{+2,\,k}, \tag{76}$$

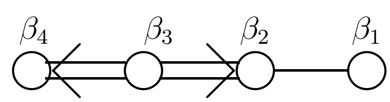
and we obtain the same Dynkin diagram:



where in the second construction the roles of α_{+1} and α_6 are exchanged. Let's observe that many non-standard (simply-laced) Dynkin diagrams can be folded to obtain other (non simply-laced) Dynkin diagrams. For example, in the case of E_6, we can identify the roots α_1 and α_2 with α_5 and α_4 respectively (so the new simple roots are $\beta_1 = \alpha_1 + \alpha_5$, $\beta_2 = \alpha_2 + \alpha_4$ and $\beta_i = 2\,\alpha_i$ for $i = 3, +1, +2, +3$), obtaining the following folded Dynkin diagram:



Actually, if we consider only the first extension of E_6, then we can identify also α_{+1} with α_6 and we obtain:



4.3 E_7

In this case we could use the fundamental weights Λ_1, Λ_2, Λ_3 and Λ_5. To illustrate the procedure, let's consider the weight Λ_5 $((\Lambda_5, \Lambda_5) = 4)$ and add the simple roots:

$$\alpha_{+1} := -\Lambda_5 + k_+ - k_-, \qquad \alpha_{+1}^2 = 2, \qquad (\alpha_{+1}, \alpha_i) = -\delta_{5,\,i}, \tag{77}$$

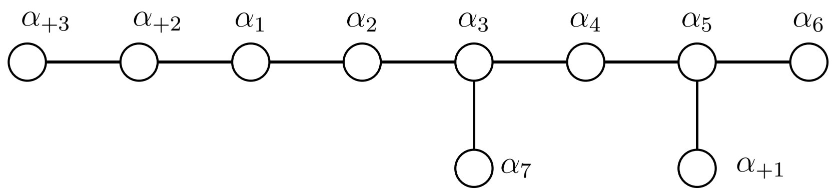
with

$$\Lambda_5 = \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8, \qquad \Lambda_5^2 = 4. \tag{78}$$

Then

$$\alpha_{+2} := -h.r. - 2k_- - l_- = \varepsilon_7 - \varepsilon_8 - 2k_- - l_- \qquad \alpha_{+3} = l_+ + l_- \,, \tag{79}$$

with $(\alpha_{+2}, \alpha_k) = -\delta_{k,\,1}$ and $(\alpha_{+3}, \alpha_j) = -\delta_{j,\,+2}$. In this way we obtain the Dynkin diagram:



Let us call $\widehat{\Lambda}_i := -\Lambda_i + k_+ - ak_-$, where $a \in \mathbb{Z}_+$ and Λ_i , $i = 1, 2, 3, 5$, is a fundamental weight. We have

$$\widehat{\Lambda}_i^2 = 2 \qquad (\widehat{\Lambda}_3, \widehat{\Lambda}_5) = 0 \tag{80}$$

4.4 E_8

E_8 root lattice is self-dual, so it coincides with the weight lattice. The non-standard extension can be made adding to the simple root system a root equal to the opposite of any weight Λ_i $(i = 1, \ldots, 8)$ plus some combination of k_+ and k_-. In this way we have 8 different extensions of E_8, with the nodes $+2$, $+3$ always in the same position (that is $(\alpha_{+2}, \alpha_7) = -1$ and $(\alpha_{+3}, \alpha_{+2}) = -1$), while the node $+1$ moves from the node 1 to the node 8, when i runs from 1 to 8 respectively. Actually, this situation is general for the non-standard extensions.

Let us emphasize again that the choice of the extended simple roots is not unique at all. Motivated by this consideration, we observe that classically we have $E_6 \subset E_7 \subset E_8$, while this inclusion is lost when we consider the corresponding affine algebras (and the same thing is true for the double and the triple extensions). So we look for an algebra that may contain all the E series and the $E^{(1)}$ series. This is possible considering the following non-standard extension of E_8 (in which the new simple roots are linked to those of E_8 using different fundamental weights). We add the new simple root:

$$\alpha_{+1} := -h.r. + k_+ + l_+ = -(\varepsilon_7 + \varepsilon_8) + k_+ + l_+, \quad \alpha_{+1}^2 = 2, \tag{81}$$

Then we add the two simple roots:

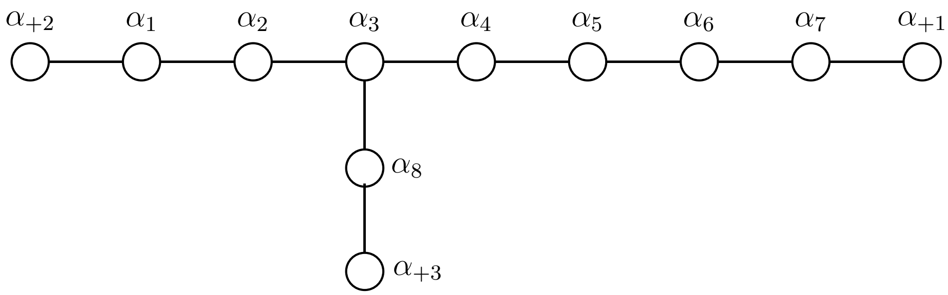
$$\Lambda_1 = 2\varepsilon_8, \qquad \Lambda_8 = \frac{1}{2}\left(\sum_{i=1}^{7} \varepsilon_i + 5\varepsilon_8\right), \tag{82}$$

$$\alpha_{+2} := -\Lambda_1 - k_- + l_+ - l_-, \quad \alpha_{+1}^2 = 2, \tag{83}$$

and

$$\alpha_{+3} := -\Lambda_8 + k_+ + l_+ - 3l_-, \quad \alpha_{+3}^2 = 2, \tag{84}$$

so that the only non-zero scalar products are: $(\alpha_{+1}, \alpha_7) = -1$, $(\alpha_{+2}, \alpha_1) = -1$ and $(\alpha_{+3}, \alpha_8) = -1$ and we have the following Dynkin diagram:



This algebra contains E_8 (then also E_7 and E_6) and all the affinizations $E_{6,7,8}^{(1)}$, so it seems to be interesting for its content in sub-algebras. Let us call $\widehat{\Lambda}_i = -\Lambda_i + k_+ - ak_-$, where $a \in \mathbb{Z}_+$ and Λ_i, is any fundamental weight. We have $(\widehat{\Lambda}_i^2 = 2)$

$$\begin{aligned} & (\widehat{\Lambda}_1, \widehat{\Lambda}_2) = (\widehat{\Lambda}_1, \widehat{\Lambda}_5) = 0 \\ (\widehat{\Lambda}_4, \widehat{\Lambda}_8) = 0 \quad & (\widehat{\Lambda}_2, \widehat{\Lambda}_3) = (\widehat{\Lambda}_2, \widehat{\Lambda}_4) = (\widehat{\Lambda}_2, \widehat{\Lambda}_5) = (\widehat{\Lambda}_2, \widehat{\Lambda}_6) = 0 \\ & (\widehat{\Lambda}_2, \widehat{\Lambda}_3) = (\widehat{\Lambda}_2, \widehat{\Lambda}_4) = (\widehat{\Lambda}_2, \widehat{\Lambda}_5) = (\widehat{\Lambda}_2, \widehat{\Lambda}_6) = 0 \end{aligned} \tag{85}$$

We have proposed a procedure to build non standard triple extended Lie algebras, which we have illustrated with a number of relevant examples. As already recalled, Kac-Moody or Borcherds extensions can be defined for these Lie algebras too. On the light of the remarks of [12], it is natural to wonder if these or some of these algebras are not really subalgebras of the standard triple extended Lie algebras. This point will be discussed in the next section, where we discuss also a few examples of subalgebras which point out the intriguing and surprising structure of the subalgebras.

5 Subalgebras of extended Lie algebras

First of all, let us discuss another non-standard procedure to extend a Lie algebras of rank r. If one adds to the simple roots of $\mathcal{G}$ the opposite of the h.r. $\alpha_0 \equiv \alpha_{r+1}$ and, to recover the linear independence of the simple root system, one glues the light like vector k_+ to a simple root α_i $(1 \le i \le r)$, one gets exactly the affine $\mathcal{G}$. As next step one adds the new root, which belongs to $\mathrm{II}_k^{1,1}$, $\alpha_{r+2} = -(k_+ + k_-)$. From the Feingold-Nicolai theorem [12], it is easy to realize that in this way one obtains a generalized Kac-Moody algebra which is really a subalgebra of the standard overextended Lie algebra $\mathcal{G}^{++}$. Things may be different if one considers simple root systems, obtained by analogous procedure, in the lattice $\Lambda_{\mathcal{G}} \oplus \mathrm{II}_k^{1,1} \oplus \mathrm{II}_l^{1,1}$. Indeed indefinite Kac-Moody algebras are obtained which, in general, are described by Dynkin-Kac diagrams not equivalent to the ones obtained by the standard and not standard procedure described in the previous section. A general discussion of these algebras is beyond the aim of this paper and we limit ourselves to state a few properties and to present some examples. Let us start with the following

Proposition 2 *The roots $\widehat{\Lambda}_i$ defined in Sec. 4 are roots of the standard overextended $\mathcal{G}^{++}$.*

Proof: We shall explicitly write $\widehat{\Lambda}_i$ in terms of the simple roots of $\mathcal{G}^{++}$. Let us remark that, by construction, $\mathcal{G}^{++}$ contains two affine $\mathcal{G}$, i.e. $\mathcal{G}^{+}$, whose real root system is formed by the roots of $\mathcal{G}$ plus $n\,k_+$, respectively $n\,k_-$, ($n \in \mathbf{Z}$). Clearly the following decomposition in roots is not all unique.

1. $so(2N)$

$$\begin{aligned} \widehat{\Lambda}_{2m} &= -(-\varepsilon_1-\varepsilon_2+k_+)+(-\varepsilon_3-\varepsilon_4-k_-)+\ldots+(-\varepsilon_{2m-1}-\varepsilon_{2m}-k_-) \\ &= -\Lambda_{2m}+k_+-(m-1)\,k_- \end{aligned} \tag{86}$$

2. E_6

$$\begin{aligned} \widehat{\Lambda}_3 &= \frac{1}{2}[-\sum_{i\neq 6,7}\varepsilon_i+\varepsilon_6+\varepsilon_7+k_+]+\frac{1}{2}[\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7-\varepsilon_8-2k_-] \\ &= -\Lambda_3+k_+-2\,k_- \end{aligned} \tag{87}$$

3. E_7

$$\begin{aligned} \widehat{\Lambda}_2 &= \frac{1}{2}[\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6+\varepsilon_7-\varepsilon_8+k_+]+(\varepsilon_7-\varepsilon_8-2k_-) \\ &= -\Lambda_2+k_+-2k_- \end{aligned} \tag{88}$$

$$\begin{aligned} \widehat{\Lambda}_3 &= (-\varepsilon_3+\varepsilon_7+k_+)+(-\varepsilon_4+\varepsilon_7-2k_-)+(-\varepsilon_5-\varepsilon_8-k_-)+(-\varepsilon_6-\varepsilon_8-2k_-) \\ &= -\Lambda_3+k_+-5k_- \end{aligned} \tag{89}$$

$$\widehat{\Lambda}_5 = (-\varepsilon_3-\varepsilon_6+k_+)+(-\varepsilon_7+\varepsilon_8-k_-) = -\Lambda_5+k_+-k_- \tag{90}$$

4. E_8

$$\begin{aligned} \widehat{\Lambda}_1 &= (-\varepsilon_i-\varepsilon_8+k_+)+(\varepsilon_i-\varepsilon_8-k_-) = -\Lambda_1+k_+-k_- \quad (i\neq 8) \\ \widehat{\Lambda}_2 &= \frac{1}{2}[(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7-\varepsilon_8)+k_+]+(-\varepsilon_3-\varepsilon_8-k_-) \\ &+ (-\varepsilon_4-\varepsilon_8-2k_-)+(-\varepsilon_5-\varepsilon_8-3k_-) = -\Lambda_2+k_+-6k_- \\ \widehat{\Lambda}_3 &= (-\varepsilon_3-\varepsilon_8+k_+)+(-\varepsilon_4-\varepsilon_8-2k_-)+(-\varepsilon_5-\varepsilon_8-3k_-) \\ &+ (-\varepsilon_6-\varepsilon_8-4k_-)+(-\varepsilon_7-\varepsilon_8-5k_-) = -\Lambda_3+k_+-14k_- \\ \widehat{\Lambda}_4 &= (-\varepsilon_4-\varepsilon_8+k_+)+(-\varepsilon_5-\varepsilon_8-2k_-)+(-\varepsilon_6-\varepsilon_8-3k_-)+(-\varepsilon_7-\varepsilon_8-4k_-) \\ &= -\Lambda_4+k_+-9k_- \\ \widehat{\Lambda}_5 &= (-\varepsilon_5-\varepsilon_8+k_+)+(-\varepsilon_6-\varepsilon_8-2k_-)+(-\varepsilon_7-\varepsilon_8-3k_-) \\ &= -\Lambda_5+k_+-5k_- \\ \widehat{\Lambda}_6 &= (-\varepsilon_6-\varepsilon_8+k_+)+(-\varepsilon_7-\varepsilon_8-2k_-) = -\Lambda_6+k_+-2k_- \\ \widehat{\Lambda}_8 &= \frac{1}{2}[(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6+\varepsilon_7+\varepsilon_8)+k_+]+(-\varepsilon_6-\varepsilon_8-k_-) \\ &+ (-\varepsilon_7-\varepsilon_8-2k_-) = -\Lambda_8+k_+-3k_- \end{aligned} \tag{91}$$

Therefore, the considered combinations of the fundamental weights with the vectors of the Lorentzian lattices $\mathrm{II}_k^{1,1}$ do belong to the root system of the overextended Lie algebras, even if the considered fundamental weights do not belong to the root system of the native Lie algebra [8]. It follows that the double non standard extensions (called previously *affine extensions*) are all subalgebras, of the same rank, of $\mathcal{G}^{++}$. Indeed it is easy to verify that the differences $\beta_i - \beta_j$ $(\forall i, j;\ \ i \neq j)$ of the simple roots β_i of the non standard double extension of $\mathcal{G}$ do not belong to the root system , therefore satisfying the conditions of the Feingold-Nicolai theorem. In particular we have

Proposition 3 *The indefinite Kac-Moody algebras of rank 10 described by the Dynkin-Kac diagrams, obtained by adding to the diagram of the affine algebra E_8, i.e. E_9, a dot, connected with a simple link to the j-th dot of E_9 $(1 \leq j \leq 9)$, is a subalgebra of E_{10}.*

The simple root systems of these subalgebra is formed by the simple roots of E_8, by a root $\widehat{\Lambda}_i$ given by eq.(91) and by the root $-\theta + ak_-$ where a is a positive integer such that $(\widehat{\Lambda}_i,\ -\theta + ak_-) = 0$.

One can naturally ask if an analogous theorem holds for E_{11}, that is if the triple non standard extensions of E_8 form Lorentzian algebras of rank 11, subalgebra of E_{11}. We have:

Proposition 4 *The indefinite Kac-Moody algebras of rank 11 obtained by adding to the simple root system of the algebra E_8 three roots: $\alpha_{+1} = \widehat{\Lambda}_j$, connected with a simple link to the j-th dot of E_8 $(1 \leq j \leq 8,\ j \neq 7)$; $\alpha_{+2} = -\theta - ak_-$, where a is a positive integer such that $(\alpha_{+2}\, \alpha_{+1}) = 0$ and $\alpha_{+3} = l_+ + l_- - k_-$, simply linked with α_{+1}, is a subalgebra of E_{11}.*

The proof is straightforward using the explicit expressions of $\widehat{\Lambda}_j$, given in eq.(91). Let us remark that these subalgebras do not have as subalgebra E_{10}. Below we give the simple roots systems of a set (not exhaustive) of rank eleven subalgebra E_{11}, which contains as subalgebra E_{10}. Let us consider the following simple root system of E_{11}: α_i $(1 \leq i \leq 8)$ are the simple roots of E_8, $\alpha_9 = -\alpha_0 - k_+$, $\alpha_{10} = k_+ + k_-$ and $\alpha_{11} = (l_+ + l_-) - k_+$. We follow the convention of [13] and, for the reader convenience, we explicitly write here the E_8 simple root system and the root system Δ:

$$\begin{gathered} \alpha_1 = \tfrac{1}{2}\left(\varepsilon_1 + \varepsilon_8 - \textstyle\sum_{j=2}^{7} \varepsilon_j\right) \qquad \alpha_i = \varepsilon_i - \varepsilon_{i-1} \quad i = 2, \ldots, 7 \\ \alpha_8 = \varepsilon_1 + \varepsilon_2 \qquad h.r. := \alpha_0 = \varepsilon_7 + \varepsilon_8 \end{gathered} \tag{92}$$

$$\Delta = \left\{ \frac{1}{2}\left(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \pm \varepsilon_8\right),\ \pm\varepsilon_i \pm \varepsilon_j \right\} \tag{93}$$

where the total number of + signs (or − signs) in the first expression is an even number. Let us consider the E_{10} Dynkin-Kac diagram obtained by the E_{11} diagram, deleting the dot corresponding to 11-th simple root. Let us denote by $E_{10}^{(j)}$ the algebra of rank 11 whose Dynkin-Kac diagram is obtained by the E_{10} diagram adding a dot (in the following denoted by +1) with a simple link to the j-th dot $(1 \leq j \leq 10)$ and by $\beta_i^{(j)}$ $(i = 1, \ldots, 10, +1)$ the simple roots of $E_{10}^{(j)}$. In the following we do not explicitly write the upper label j in the roots. Clearly $E_{10}^{(10)} = E_{11}$ and, in this case, $\beta_{+1} = \alpha_{11}$. We make the following (not unique) choice for the simple root system of $E_{10}^{(j)}$ $(1 \leq j \leq 9)$:

[8] One can recover the generators by the change $\alpha \to E_\alpha$ and $\alpha + \beta \to [E_\alpha,\, E_\beta]$.

- j = 9)

$$\beta_1 = \alpha_1 - k_+ \quad \beta_i = \alpha_i \quad i = 2, \ldots, 7, 8$$
$$\beta_9 = -\varepsilon_7 + \varepsilon_8 + k_- \quad \beta_{10} = -\alpha_0 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{94}$$

- j = 7)

$$\beta_i = \alpha_i \quad i = 1, \ldots, 6, 8 \quad \beta_7 = \alpha_7 - k_+$$
$$\beta_9 = \alpha_{10} \quad \beta_{10} = \alpha_{11} \quad \beta_{+1} = -\alpha_0 \tag{95}$$

- j = 6)

$$\beta_i = \alpha_i \quad i = 1, \ldots, 5, 7, 8 \quad \beta_6 = \alpha_6 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = \varepsilon_7 + \varepsilon_6 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{96}$$

- j = 5)

$$\beta_i = \alpha_i \quad i = 1, 2, 3, 4, 6, 7, 8 \quad \beta_5 = \alpha_5 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = \varepsilon_5 + \varepsilon_8 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{97}$$

- j = 4)

$$\beta_i = \alpha_i \quad i = 1, 2, 3, 5, 6, 7, 8 \quad \beta_4 = \alpha_4 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = \varepsilon_4 + \varepsilon_8 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{98}$$

- j = 3)

$$\beta_i = \alpha_i \quad i = 1, 2, \ldots, 7, 8 \quad \beta_3 = \alpha_3 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = \varepsilon_3 + \varepsilon_8 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{99}$$

- j = 2)

$$\beta_i = \alpha_i \quad i = 1, 3, \ldots, 7, 8 \quad \beta_2 = \alpha_2 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = -\varepsilon_1 + \varepsilon_8 - k_+ \quad \beta_{+1} = \alpha_{11} \tag{100}$$

- j = 1)

$$\beta_1 = -\tfrac{1}{2} \textstyle\sum_{i=i}^{8} \varepsilon_i + k_- \quad \beta_2 = \alpha_8 \quad \beta_i = \alpha_i \quad i = 3, \ldots, 6 \quad \beta_8 = \alpha_2$$
$$\beta_7 = \alpha_7 \quad \beta_9 = -\varepsilon_7 + \varepsilon_8 \quad \beta_{10} = -\alpha_1 + k_+ \quad \beta_{+1} = \alpha_{11} \tag{101}$$

- j = 8)

$$\beta_i = \alpha_i \quad i = 1, \ldots, 7 \quad \beta_8 = \alpha_8 + k_-$$
$$\beta_9 = -\alpha_0 \quad \beta_{10} = \tfrac{1}{2} \textstyle\sum_{i=1}^{8} \varepsilon_i - k_+ \quad \beta_{+1} = \alpha_{11} \tag{102}$$

It is easy to verify that the roots β_i belong to the roots sistems of E_{11},[9] while the differences $\beta_i - \beta_j$ ($\forall i, j;\ i \neq j$) ($|\beta_i - \beta_j|^2 \geq 4$) do not belong, therefore satisfying the conditions of the Feingold-Nicolai theorem. The Dynkin-Kac diagrams describing these algebras contain loops except for $j = 7$. This algebra has been considered in [14], where it has been denoted EE_{11}, and it has been shown to be a subalgebra of E_{11} by explicitly constructing the generators by commutation of the E_{11} generators. The Dynkin-Kac diagrams for these algebras are easily drawn by adding to the E_{10} diagram a dot simply connected with the j-th dot and then connecting, with a simple link, the following dots: $j = 9)\ \ 7-10$; $j = 6)\ \ 1-10$; $j = 5)\ \ 6-10$; $j = 4)\ \ 5-10$; $j = 3)\ \ 4-10$; $j = 2)\ \ 8-10$; $j = 1)\ \ 7-10, 8-10$; $j = 8)\ \ 1-10$. Of course, these subalgebras do not exhaust the set of 11 dimensional indefinite Kac-Moody subalgebras of E_{11}. In particular we have not considered the subalgebras which do appear as invariant algebras with respect to an involution of the generators of E_{11}, see [14] and [15].

6 Conclusions and future developments

In studying 4-extended Lie algebras, we have seen that Borcherds algebras seem to emerge naturally. This remark rises the question: which are the fingerprints of a theory which exhibits a symmetry under a Borcherds algebra? This question is indeed interesting on the light of the remark that many dualities have a group-theoretical origin in the Weyl group of the algebra. The Weyl group of the Borcherds algebra has peculiar properties as the reflection with respect to the imaginary vanishing roots is not defined. Some particular properties related to this kind of algebras have already been discussed in [16]. The non-standard extension introduced in this paper have peculiar features, which deserve further investigation, on both their mathematical structure and their possible physical relevance. A classification of these algebras is beyond the aim of this paper, where we present only a few representative examples. As, however, very little is known on Lorentzian Kac-Moody algebras, we believe that any new information is interesting. In the cited literature on the physical role of the very extended Lie algebras, non-linear realizations of the indefinite Kac-Moody algebras are used. How a Chevalley realization of this algebra looks like? In [17] a procedure to build up vertex realization of Lorentzian algebra with only a lattice $\mathrm{II}^{1,1}$ has been proposed and applied to the very simple case of the overextended A_1 algebra. It seems possible to generalize that procedure to the triple extended Lie algebras. Moreover, it has also been argued by P. West [18] that $sl(32)$ is contained in the Cartan invariant sub-algebra of E_{11}. At first sight the rank of $sl(32)$ is too large to be a sub-algebra, so it seems that very extended algebras, at least in the non linear realization, admit finite dimensional sub-algebras which naively could not be there. The investigation of the finite Lie subalgebras of the indefinite Kac-Moody algebras requires new methods beyond the very familiar ones used in the case of finite Lie algebra, which are essentially based on the Dynkin methods. This feature is not completely unrelated with the property, noted in [12], that the set of infinite dimensional sub-algebras of Lorentzian algebras is quite rich and surprising. We have illustrated this feature in Sec. 5, discussing a class of subalgebras of E_{11}, but it would be useful to dispose of techniques to build up explicitly or to identify classes of these sub-algebras or to dispose of further examples.

[9] Actually they belong, except for α_{11}, to the root system of affine $E_8 = E_9$ which is a regular sub-algebra of E_{11}

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A Some facts about the lattice $\mathrm{II}^{1,1}$

We review some basic facts about the lattice $\mathrm{II}^{1,1}$, which is the only Lorentzian even self-dual lattice in dimension 2. The points in this lattice can be described as the vectors:

$$(n, m) \tag{103}$$

with $n, m \in \mathbb{Z}$ and Gram matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, with eigenvalues ± 1. In this way, the scalar product between two vectors $a = (a_+, a_-)$ and $b = (b_+, b_-)$ can be written as: $a \cdot b = -a_+ b_- - a_- b_+$. We can take $k_+ \equiv (1, 0)$ and $k_- \equiv (0, -1)$ as basis vectors in $\mathrm{II}^{1,1}$; with this choice, we have:

$$k_\pm \cdot k_\pm = 0\,, \quad k_\pm \cdot k_\mp = 1. \tag{104}$$

All the vectors in $\mathrm{II}^{1,1}$ can be written as $v = p\, k_+ + q\, k_-$ (with $p, q \in \mathbb{Z}$); in particular there are only two vectors of squared norm 2, $\pm(k_+ + k_-)$, but infinite vectors of positive (≥ 4) and negative (≤ -2) squared norm, being $v^2 = 2p\,q$.

B Definition of Borcherds Algebras

The best way to think of a Borcherds algebra is to consider it as a generalization of a finite-dimensional simple Lie algebra. The definition is based on the Serre-Chevalley construction of finite-dimensional algebras; we follow [11].These algebras always have a symmetric matrix and their structure is very similar to that of ordinary Kac-Moody algebras, the only major difference is that generalized Kac-Moody algebras allow the presence of imaginary simple roots. Let $A = (a_{i,j})$ a $n \times n$ (real) symmetric matrix satisfying the following properties:

- $a_{ii} = 2$ or $a_{ii} \leq 0$,
- $a_{ij} \leq 0$ if $i \neq j$,
- $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.

Then the Borcherds algebra $\mathcal{G}(A)$ associated with the Cartan matrix A is the Lie algebra given by the following generators and relations.
$3n$ *Generators*: e_i, f_i and h_i
Relations:

- $[h_i, h_j] = 0$,
- $[e_i, f_j] = \delta_{ij} h_i$,
- $[h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} e_j$,
- $e_{ij} := (\mathrm{ad}\ e_i)^{1-a_{ij}}\, e_j = 0$, $f_{ij} := (\mathrm{ad}\ f_i)^{1-a_{ij}}\, f_j = 0$ if $a_{ii} = 2$ and $i \neq j$,

- $e_{ij} := [e_i, e_j] = 0$, $f_{ij} := [f_i, f_j] = 0$ if $a_{ii} \leq 0$, $a_{jj} \leq 0$ and $a_{ij} = 0$.

The elements h_i form a basis for an abelian subalgebra of $\mathcal{G}(A)$, called Cartan subalgebra $\mathcal{H}(A)$; as it happens for Kac-Moody algebras, $\mathcal{G}(A)$ has the triangular decomposition:

$$\mathcal{G}(A) = \mathcal{N}_- \oplus \mathcal{H}(A) \oplus \mathcal{N}_+ \tag{105}$$

and has many of the properties of the usual Kac-Moody algebras (real and imaginary roots, etc.). In particular, in this paper, we have considered Borcherds algebras with just one imaginary simple root (with squared norm 0), which we have added by hand. A rank 2 Borcherds algebra in the lattice $\mathrm{II}^{1,1}$ can be constructed as follows: the Cartan matrix is given by

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \tag{106}$$

A possible choice for the simple roots is

$$\alpha_1 = k_+ \,, \quad \alpha_2 = -(k_+ + k_-) \,, \tag{107}$$

with Weyl vector $\rho = -k_+$ (defined by $(\rho,\, \alpha_i) = 1/2\ (\alpha_i,\, \alpha_i)$).

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